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MATERIAL INSTABILITIES OF AN INCOMPRESSIBLE ELASTIC CUBE UNDER TRIAXIAL TENSION

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Abstract—Besides purely geometrical non-linearity, unstable solutions in finite elasticity may arise due to non-linear material behaviour. In this paper, a stability criterion is developed with which these so-called material instabilities can be distinguished from geometrical or structural instabilities like the buckling of shells and plates. Finally, the value of this criterion is demonstrated analytically by examining the case of a cube under triaxial tension. To obtain realistic results, the stability investigation is based on the material model of Ogden, which is characterized by excellent agreement with experimental results. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Instabilities in finite elasticity can be caused by geometrically non-linear effects as well as by non-linear material behaviour. Examples of the former type of instability, so-called structural instability, include buckling in shells and plates. They occur when negative stresses dominate in the body. To detect this kind of instability, large deformations must be considered, whereas the material behaviour can still be linear. The geometry of the system and the boundary conditions influence such instability behaviour qualitatively, while material properties have only a quantitative influence. That such is the case can be seen from the classical calculations of the buckling loads of rods, plates and shells [see Chajes (1974); Thompson and Hunt (1984)]. Indeed, for fixed geometry and boundary conditions, the choice of different material properties may increase or decrease the buckling load, but cannot prevent the buckling of the structure.

In contrast to that, the material stability behaviour is substantially dominated by the type of the material model and the choice of the material parameters, i.e. material instabilities can be avoided by choosing a different material. Certainly, the boundary conditions and the geometry of the structure still have an influence on the stability behaviour. In general, then, the overall stability of a structure is influenced by its geometry, the boundary conditions it is subject to, as well as its material behaviour.

On the other hand, material instability is in fact of a completely different nature than a structural instability. First of all, the secondary deformation state can still be homogeneous, in contrast to buckled structures. Furthermore, material instabilities arise only when the stress field in the body is positive. Finally, large, rather than moderate, deformations have to be applied to induce material instability. An example of this is a quadratic sheet in a plane stress state with equal loads on each side [see Shield (1971); Ogden (1985); Kearsley (1986); Chen (1987); Müller (1992)]. In the context of the Mooney–Rivlin model, beyond a certain load level, the quadratic deformation state (same strains in both directions) becomes unstable and a secondary load–deformation path associated with a rectangular deformation state is obtained. For the neo-Hooke model, however, the quadratic deformation state remains stable.

The question is now whether this effect, which represents a typical material instability, is an artifact of the material model, or whether it is physical. Such an instability was observed in an experiment of Treloar (1948), carried out for a swollen vulcanized rubber with 2% sulphur. According to Treloar, in contrast to the neo-Hooke model, the Mooney–Rivlin model fits the experimental results in a satisfactory way. Although the kind of rubber used for the experiment exhibits noticeable hysteresis effects (Treloar, 1947); Treloar (1948) excluded the possibility that this instability is caused by relaxation. Interestingly enough, such an instability was not reported in any of more recent experimental studies (Obata *et al.*, 1970; James *et al.*, 1975; Jones and Treloar, 1975; Vangerko and Treloar, 1978). Taking a closer look at the experimental conditions in these works, however, reveals that Treloar (1948) was the only experimentalist to control the force in both directions, i.e. the other experimentalists controlled the displacement in at most one direction, which prevented them from observing the instability.

In practical situations, of course, it is desirable to avoid instabilities of any kind. Investigation of material or other instabilities detected in experimental work via material modelling may allow one to determine how such instabilities can be avoided. Furthermore, the detection of material instabilities during modelling which have not been observed experimentally may be an indication that the material properties of the model are not in complete agreement with its physical behaviour and may require further investigation.

The purpose of the current work is to present a method allowing the detection of material instabilities, and their differentiation from structural instabilities, in a simple way. The method is general enough to include visco-elastic and plastic material behaviour, which is exhibited by rubber as well. To allow analytical calculations, however, the stability investigations in this paper are restricted to finite elasticity.

To avoid material instabilities, the dependence of these on material properties must be described as realistically as possible. In most of the earlier contributions [see, besides the above mentioned, Rivlin (1974); Sawyers (1976); Ball and Schaeffer (1983)], only two-parameter models like the Mooney–Rivlin model (Mooney, 1940; Rivlin, 1948 a–d; 1949 a–c; Rivlin and Saunders 1951), or one-parameter models like the neo-Hooke model (Treloar, 1943), were used. For most rubbers, these material models are too simple to explain experimental results (Tschoegl, 1971; James *et al.*, 1975; Jones and Treloar, 1975; Twizell and Ogden, 1983). On this basis, the present work is based on a more general material model for finite elastic material behaviour, i.e. the six-parameter model of (Ogden 1972, 1984). The fit of this material model to experimental data is also discussed in detail by Twizell and Ogden (1983), and shows very good agreement with the experimental results of Treloar (1944) up to 600% strain. Since the material model is purely elastic, the fit was carried out for a type of rubber [see Treloar (1944)] which exhibits nearly ideally elastic deformation behaviour (Ogden, 1972). Later, James *et al.* (1975) and Jones and Treloar (1975) confirmed the applicability of Ogden's material model to rubbers showing noticeable hysteresis. The material model of Ogden is convenient as a basis for the present stability investigations because it contains the neo-Hooke and the Mooney–Rivlin model as special cases. Thus, the agreement with previous results (Rivlin, 1974; Sawyers, 1976; Ball and Schaeffer, 1983) can be checked easily. The method of eigenvalue splitting described in this paper, however, is not restricted to the use of the Ogden model and can be applied to other material models without much effort.

To begin, we introduce the strain energy function and some restrictions on the material parameters guaranteeing strong ellipticity of the resulting equations of motion in Section 2. For the purpose of detailed parameter studies, the stability investigations are carried out analytically. Due to the complexity of the non-linear equations, however, it is necessary to restrict the calculation to homogeneous deformation states. A corresponding "local" stability criterion is developed in Section 3. Inhomogeneous deformation states can be taken into account if a numerical method, e.g. the finite element method, is used [see Reese (1994), Reese and Wriggers (1995)].

Although it is well-known that there are different types of instability arising in finite elasticity, none of the earlier investigations dealt with a method to distinguish one type of instability from another. For this purpose, a new method called eigenvalue splitting is formulated in Section 4, with which one can distinguish material instabilities from structural instabilities. Finally, in Section 5, the example of a cube under triaxial tension is investigated. This example has been treated in the literature before [see Rivlin, 1974; Sawyers, 1976; Ball and Schaeffer, 1983; Marsden and Hughes, 1983]. Since these authors only considered the Mooney–Rivlin material, their results are obtained here as special cases.

2. STRAIN ENERGY FUNCTION

According to the principle of objectivity and the assumption of material isotropy the specific strain energy function W is written as a function of the principal stretches λ_1, λ_2 and λ_3 [see Ogden (1972)]. These represent the principal values of the right stretch tensor $\mathbf{U} = \mathbf{R}^T \cdot \mathbf{F}$ and the left stretch tensor $\mathbf{V} = \mathbf{F} \cdot \mathbf{R}^T$ with the orthogonal tensor $\mathbf{R}$ and the deformation gradient $\mathbf{F}$. To fulfill the incompressibility condition exactly, the Lagrange multiplier method is applied. We use the strain energy function

$$W = \sum_{r=1}^{n} \frac{\mu_r}{\alpha_r} (\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3) \tag{1}$$

derived by Ogden (1972) for incompressible materials. W depends on the material parameters μ_r and α_r.

The strain energy function is constructed in such a way that if fulfills the polyconvexity condition [see Ball (1977)] and some requirements for existence of solutions [see Ciarlet (1988); Simo (1995)]. The requirements are summarized as

$$\lim_{J \to 0} W = \infty,$$

$$\lim_{J \to \infty} W = \infty,$$

$$\lim_{\mathbf{F} \to \mathbf{1}} W = 0.$$

$J = \lambda_2 \lambda_2 \lambda_3$ is the determinant of the deformation gradient $\mathbf{F}$. To fulfill the condition of polyconvexity, the inequalities

$$\mu_r \alpha_r > 0,$$

$$|\alpha_r| > 1$$

must hold for the material parameters (no summation over r) [see Ogden (1972, 1985); Ciarlet (1988)]. W represents different materials according to the choice of the parameters. For a neo-Hooke material ($n = 1, \alpha_1 = 2$) [see Treloar (1943)], μ_1 can be identified with the shear modulus G, while for a Mooney–Rivlin-material ($n = 2, \alpha_1 = 2, \alpha_2 = -2$) [see Mooney (1940); Rivlin (1848 a–d); (1949 a–c); Rivlin and Saunders (1951)] $2G = \mu_1 \alpha_2$ holds.

In contrast to the neo-Hooke model, the six-parameter model developed by Ogden (1972) correlates with experimental results even for large strains up to 600%. The material parameters determined by Ogden fulfill all mathematical and physical requirements. Additionally, as can be seen above, the six-parameter model contains simpler common material models like the neo-Hooke model and the Mooney–Rivlin model as special cases.

3. STABILITY CRITERION FOR PURE HOMOGENEOUS DEFORMATIONS

We start from the weak form of equilibrium (p Lagrange parameter)

$$\hat{g}(\mathbf{F},p) = \delta\left[\int_{\mathscr{B}_0}(W - p\hat{U}(J)\mathrm{d}V\right] - g_a = 0, \tag{2}$$

where the Lagrange parameter p has been introduced for the incompressibility condition $\hat{U}(J) = J-1 = 0$. From eqn (2), the two equations

$$\hat{g}_F(\mathbf{F},p) = \int_{\mathscr{B}_0}\mathbf{P}:\delta\mathbf{F}\mathrm{d}V - g_a = 0,$$

$$\hat{g}_p(\mathbf{F},p) = \int_{\mathscr{B}_0} -\hat{U}(J)\delta p\mathrm{d}V = 0 \tag{3}$$

can be derived. A colon stands for the scalar product of two tensors. For the first Piola–Kirchhoff stress tensor $\mathbf{P}$ we have the expression

$$\mathbf{P} = \hat{\mathbf{P}}(\mathbf{F},p) = \frac{\partial W}{\partial \mathbf{F}} - p\frac{\partial U}{\partial \mathbf{F}}$$

g_a denotes the virtual work of the static external load, which is here assumed as being deformation independent.

From the Taylor expansion of $g_F = \hat{g}_F(\mathbf{F},p)$ we obtain with $\mathbf{F} = \bar{\mathbf{F}} + \Delta\mathbf{F}$ and constant p

$$\hat{g}_F(\mathbf{F},p) - \hat{g}_F(\bar{\mathbf{F}},p) = D\hat{g}_F(\bar{\mathbf{F}},p):\Delta\mathbf{F} + | \ldots, \tag{4}$$

where

$$D\hat{g}_F(\bar{\mathbf{F}},p):\Delta\mathbf{F}: = \frac{\mathrm{d}}{\mathrm{d}\alpha}\hat{g}_F(\bar{\mathbf{F}} + \alpha\Delta\mathbf{F},p)\,|_{\alpha=0}$$

defines the directional derivative (Gateaux-derivative).

The solution of eqn (2) is unique if the inequality

$$D\hat{g}_F(\mathbf{F},p):\Delta\mathbf{F} \neq 0$$

holds. In the case of a singular or non-unique solution, the directional derivative goes to zero, which yields eqn (2)

$$\int_{\mathscr{B}_0}\delta\mathbf{F}:\frac{\partial \mathbf{P}}{\partial \mathbf{F}}:\Delta\mathbf{F}\mathrm{d}V = 0. \tag{5}$$

Since only pure homogenous deformations will be investigated in the following, the stability behaviour is analyzed locally. Then, in every point of the body, the condition

$$\delta\mathbf{F}:\mathscr{A}:\Delta\mathbf{F} = 0 \tag{6}$$

has to be fulfilled, were $\mathscr{A}$ denotes the rank four constitutive tensor

$$\mathscr{A} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} - p \frac{\partial^2 U}{\partial \mathbf{F} \partial \mathbf{F}}.$$

Equivalent to eqn (6) is the solution of the eigenvalue problem

$$(\mathscr{A} - \tilde{\omega}_{(i)} \mathbf{1}^4) : \tilde{\mathbf{\Phi}}_{(i)} = 0 \tag{7}$$

with at least one zero eigenvalue $\tilde{\omega}_{(i)}$. $\tilde{\mathbf{\Phi}}_{(i)}$ represents the associated eigentensor. Note that, over bracketed indices, no summation takes place. $\mathbf{1}^4$ represents the rank four identity tensor. A stable unique solution is indicated by the positive definiteness of $\mathscr{A}$. This means for all eigenvalues

$$\boxed{\tilde{\omega}_{(i)} = \tilde{\mathbf{\Phi}}_{(i)} : \mathscr{A} : \tilde{\mathbf{\Phi}}_{(i)} > 0, \quad \tilde{\mathbf{\Phi}}_{(i)} : \mathbf{1}^4 : \tilde{\mathbf{\Phi}}_{(i)} = 1.} \tag{8}$$

In eqn (8), the symmetry of the constitutive tensor $\mathscr{A}$, which is given by the derivation of $\mathscr{A}$ from W and U, has been used. In the following, the inequality (8) will be called "local stability criterion". It is essential that eqn (8) represents a stronger requirement than the condition for strong ellipticity

$$(\mathbf{m} \otimes \mathbf{N}) : \mathscr{A} : (\mathbf{m} \otimes \mathbf{N}) > 0, \quad \mathbf{m} \neq 0, \quad \mathbf{N} \neq 0, \tag{9}$$

which is implied by the condition of polyconvexity (Ball, 1977). The sign $\otimes$ represents the dyadic product. For the difference between the requirement for stability and the one for strong ellipticity, see also Simpson and Spector (1983; 1987). Since polyconvexity can easily be guaranteed by an appropriate choice of the material parameters, loss of strong ellipticity (localization) is excluded for hyperelastic materials under static loading.

4. "MATERIAL INSTABILITIES" AND "STRUCTURE INSTABILITIES"

By means of the relation $\mathbf{P} = \mathbf{F} \cdot \mathbf{S}$ between the first Piola–Kirchhoff stress tensor $\mathbf{P}$ and the symmetric second Piola–Kirchhoff stress tensor

$$\mathbf{S} = \tilde{\mathbf{S}}(\mathbf{C}, p) = 2 \left[\frac{\partial W}{\partial \mathbf{C}} - p \frac{\partial U}{\partial \mathbf{C}} \right]$$

with the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$, we can rewrite eqn (3a) in the form

$$\tilde{g}_F(\mathbf{F}, p)' = \tilde{g}_C(\mathbf{C}, p) = \int_{\mathscr{B}_0} \mathbf{S} : \mathrm{sym}(\mathbf{F}^{\mathrm{T}} \cdot \delta \mathbf{F}) \mathrm{d}V - g_a = 0.$$

The directional derivative is calculated by

$$D\tilde{g}_C(\mathbf{C}, p) : \Delta \mathbf{C} = \int_{\mathscr{B}_0} \left[2 \frac{\partial \mathbf{S}}{\partial \mathbf{C}} : \mathrm{sym}(\mathbf{F}^{\mathrm{T}} \cdot \Delta \mathbf{F}) : \mathrm{sym}(\mathbf{F}^{\mathrm{T}} \cdot \delta \mathbf{F}) + \mathbf{S} : \mathrm{sym}(\Delta \mathbf{F}^{\mathrm{T}} \cdot \delta \mathbf{F}) \right] \mathrm{d}V.$$

Taking into account only homogeneous deformations, a singularity is indicated by

$$\mathscr{L} : \mathrm{sym}(\mathbf{F}^{\mathrm{T}} \cdot \Delta \mathbf{F}) : \mathrm{sym}(\mathbf{F}^{\mathrm{T}} \cdot \delta \mathbf{F}) + \mathbf{S} \cdot \mathrm{sym}(\Delta \mathbf{F}^{\mathrm{T}} \cdot \delta \mathbf{F}) = 0. \tag{10}$$

Here two influences have to be investigated. The first part of eqn (10) contains the rank four constitutive tensor

$$\mathscr{L} = 2\frac{\partial \mathbf{S}}{\partial \mathbf{C}} = 4\left[\frac{\partial^2 W}{\partial \mathbf{C}\partial \mathbf{C}} - p\frac{\partial^2 U}{\partial \mathbf{C}\partial \mathbf{C}}\right] \tag{11}$$

and represents, therefore, the influence of the constitutive model of the material parameters. This part can only be negative and, therefore, cause the instability of a solution, if $\mathscr{L}$ loses its positive definiteness. Note that the fulfillment of the polyconvexity condition does not require the positive definiteness of $\mathscr{A}$ or $\mathscr{L}$.

Remark 1

A frequently computed material model, which does not fulfill the condition of polyconvexity, is the so-called St. Venant material law. It is related to the strain energy function $W_V = \mu \mathbf{E}:\mathbf{E} + \lambda/2(\mathbf{E}:\mathbf{1})^2$. Although it does not match all mathematical requirements, it can still be used for the investigation of structural instabilities as buckling phenomena, when only small or moderate strains are present. This is due to the fact that instabilities of this kind are caused by the second part of eqn (10) becoming negative. Evidently, they occur independently of a special material model. Thus, the choice of the constitutive equation plays only a subordinate role, if negative stresses dominate in the body.

The split into material and structural parts is only possible if symmetric tensors are used in the weak form of equilibrium. One could object that this fact calls the physical meaning of the split into question. However, it will be shown in Section 5 that physically irrelevant eigenmodes like rigid body rotations are excluded only if the stability investigation is based on symmetric strain measures.

To obtain a local stability criterion like eqn (8), it is necessary to calculate a fourth order tensor $\mathscr{M}$ such that the equation

$$\mathbf{S}:\mathrm{sym}(\Delta\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) = \mathscr{M}:\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) \tag{12}$$

holds. By means of the relation

$$\mathbf{S}:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) = \mathscr{M}_1:(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F})$$

first the fourth order tensor $\mathscr{M}_1$ is determined. With

$$\begin{aligned}
\mathbf{S} &= S^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \\
\mathscr{M}_1 &= M_1^{abcd}\mathbf{G}_a \otimes \mathbf{G}_b \otimes \mathbf{G}_c \otimes \mathbf{G}_d \\
\mathbf{1}\cdot\mathbf{F} &= g_k \otimes \mathbf{G}^k = (\mathbf{G}^l \otimes \mathbf{G}_l)\cdot(g_k \otimes G^k) = (\mathbf{G}_l\cdot g_k)\mathbf{G}^l \otimes \mathbf{G}^k = \mathbf{F}_{lk}\mathbf{G}^l \otimes \mathbf{G}^k \\
\delta\mathbf{F} &= \delta F_{mn}\mathbf{G}^m \otimes \mathbf{G}^n
\end{aligned}$$

we obtain

$$S^{ij}\Delta F_{ki}\delta F_j^k = M_1^{ajci}F_{lc}\Delta F_i^l F_{ka}\Delta F_j^k$$

and finally

$$(S^{ij}G_{kl} - M_1^{ajci}F_{ka}F_{lc})\Delta F_i^l\delta F_j^k = 0.$$

The relation

$$G_{kl} = [(g^a \otimes g_a)\cdot G_k]\cdot[(g^c \otimes g_c)\cdot G_l) = F_{ka}F_{lc}g^{ac}$$

leads to the equation

$$(S^{ij}g^{ac} - M_1^{ajci}F_{ka}F_{lc}\Delta F^l_i \delta F^k_j = 0,$$

which has a non-trivial solution

$$M_1^{ijkl} = g^{ik}S^{jl}.$$

Analogously, we can derive tensors such that the relation

$$\begin{aligned}\mathbf{S}:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) &= \mathcal{M}_1:(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F})\\ &= \mathcal{M}_2:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}):(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F})\\ &= \mathcal{M}_3:(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):(\delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})\\ &= \mathcal{M}_4:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}):(\delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})\end{aligned}$$

holds, from which follows

$$\begin{aligned}\mathbf{S}:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) = \tfrac{1}{4}\mathcal{M}_1:(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F})+\tfrac{1}{4}\mathcal{M}_2:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}):(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F})\\ +\tfrac{1}{4}\mathcal{M}_3:(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):(\delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})+\tfrac{1}{4}\mathcal{M}_4:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}):(\delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}).\end{aligned}$$

The tensors $\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}$ and $\Delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$ are splitted in a symmetrical part sym $(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}) = \mathrm{sym}(\Delta\mathbf{F}\cdot\mathbf{F})$ and in a skew-symmetrical part skw $(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}) = -\mathrm{skw}(\Delta\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})$. We obtain

$$\mathbf{S}:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) = \tfrac{1}{4}(\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3+\mathcal{M}_4):\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) \tag{13}$$

$$+\tfrac{1}{4}(\mathcal{M}_1-\mathcal{M}_2+\mathcal{M}_3-\mathcal{M}_4):\mathrm{skw}(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) \tag{14}$$

$$+\tfrac{1}{4}(\mathcal{M}_1+\mathcal{M}_2-\mathcal{M}_3-\mathcal{M}_4):\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):\mathrm{skw}(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) \tag{15}$$

$$+\tfrac{1}{4}(\mathcal{M}_1-\mathcal{M}_2-\mathcal{M}_3+\mathcal{M}_4):\mathrm{skw}(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):\mathrm{skw}(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}). \tag{16}$$

Due to the relations

$$\begin{aligned}M_2^{abcd} &= M_1^{abdc}\\ M_3^{abcd} &= M_1^{bacd}\\ M_4^{abcd} &= M_1^{badc}\end{aligned}$$

between the coefficients of $\mathcal{M}_I(I = 1, 2, 3, 4)$ the terms (14), (15) and (16) are zero, and it remains the expression

$$\mathbf{S}:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}) = \underbrace{\tfrac{1}{4}(\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3+\mathcal{M}_4)}_{\mathcal{M}}:\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\cdot\Delta\mathbf{F}):\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\cdot\delta\mathbf{F}).$$

Thus, analogous to eqn (7), we have the eigenvalue problem

$$(\mathcal{L}+\mathcal{M}-\omega_{(i)}\mathbf{1}^4):\hat{\Phi}_{(i)} = 0.$$

The local stability criterion is now given by

$$\omega_{(i)} = \hat{\mathbf{\Phi}}_{(i)} : (\mathscr{L} + \mathscr{M} : \hat{\mathbf{\Phi}}_{(i)} > 0, \quad \hat{\mathbf{\Phi}}_{(i)} : \mathbf{1}^4 : \hat{\mathbf{\Phi}}_{(i)} = 1, \tag{17}$$

where the symmetric rank two tensor $\hat{\mathbf{\Phi}}_{(i)} = \mathrm{sym}\,(\mathbf{F}_\mathrm{T} \cdot \bar{\mathbf{\Phi}}_{(i)})$ has been introduced as eigen-tensor.

For the analysis it is convenient to carry out the calculation in the current configuration. Then the local stability criterion (17) transforms to

$$\boxed{\omega_{(i)} = \mathbf{\Phi}_{(i)} : (\mathscr{C} + \mathscr{D}) : \mathbf{\Phi}_{(i)} > 0, \quad \mathbf{\Phi}_{(i)} : \mathbf{1}^4 : \mathbf{\Phi}_{(i)} = 1} \tag{18}$$

with

$$\mathscr{C} = \frac{1}{J} L^{abcd} g_a \otimes g_b \otimes g_c \otimes g_d, \quad \mathscr{D} = \frac{1}{J} M^{abcd} g_a \otimes g_b \otimes g_c \otimes g_d \tag{19}$$

and $\mathbf{\Phi}_{(i)} = \mathbf{F}^{-\mathrm{T}} \cdot \hat{\mathbf{\Phi}}_{(i)} \cdot \mathbf{F}^{-1}$. Using the symmetry properties of $\mathscr{C}$ and $\mathscr{D}$, we obtain in matrix notation

$$\boxed{\omega_{(i)} = \hat{\Phi}_a^{(i)} \underbrace{(\hat{C}_E^{ab} + \hat{D}_E^{ab})}_{T_E^{ab}} \hat{\Phi}_b^{(i)} = \underbrace{\hat{\Phi}_a^{(i)} \hat{C}_E^{ab} \hat{\Phi}_b^{(i)}}_{\omega_{(i)}^C} + \underbrace{\hat{\Phi}_a^{(i)} \hat{D}_E^{ab} \hat{\Phi}_b^{(i)}}_{\omega_{(i)}^D}, \quad a,b = 1,2,\ldots,6.} \tag{20}$$

The explicit form of $\hat{C}_E^{ab}$ and $\hat{D}_E^{ab}$ will be derived in the next section. The index E means that the matrix elements are calculated with respect to principal axes, which coincide for the cube under triaxial tension with the symmetry axes. A material instability is associated with the negative definiteness of the material matrix $\hat{C}_E^{ab}$. In eqn (20), this is indicated by a negative eigenvalue part ω^C. In the case of $\omega^D < 0$, we have a structural instability.

This "eigenvalue splitting" is a new method to determine the reasons for instabilities in finite elasticity. For analytical investigations, the limitation to homogeneous deformations is necessary due to the complexity of the resulting equations; but the eigenvalue splitting can easily be extended to inhomogeneous deformations if a numerical method like the finite element method is used [see Reese (1994); Reese and Wriggers (1995)].

5. ANALYTICAL STABILITY INVESTIGATION OF A CUBE UNDER THREE-AXIAL TENSION

The goal of the following investigation is to analyze the role of the eigenvalue parts ω^C and ω^D for the hyperelastic cube under triaxial tension. The procedure is summarized in Table 1.

The problem of a hyperelastic cube under triaxial tension is called the "Signorini problem" in the literature. The loads are equal on each side and are equally distributed over the area. This example has been treated by Rivlin (1974); Sawyers (1976); Ball and Schaeffer (1983) [see also the textbook for Marsden and Hughes (1983)]. While the calculations of Rivlin and Sawyers are based solely on the neo-Hooke model, Ball and Schaeffer, and Marsden and Hughes also considered the Mooney–Rivlin model. However, neither of these investigations deal with the Ogden model or contain any discussion about the mechanical reason for the instabilities of the hyperelastic cube.

Before the stability investigation can take place, the coefficients $\hat{T}_E^{ij} = \hat{C}_E^{ij} + \hat{D}_E^{ij}$ have to be determined. Using the normalized orthogonal eigenvectors $\mathbf{n}_{(i)}$ of the eigenvalue problem

Table 1. Analytical stability investigation

(1) Determination of solutions which fulfill equilibrium

For every material model:

(2) Solution of the eigenvalue problem

$$\boxed{(\hat{T}_E^{ab} - \omega_{(i)}\delta^{ab})\Phi_b^{(i)} = 0} \tag{21}$$

for the load–deformation paths calculated under eqn (1)

(3) Split of the eigenvalues into material parts and structural parts with

$$\boxed{\omega(i) = \underbrace{\hat{\Phi}_a^{(i)}\hat{C}_E^{ab}\hat{\Phi}_b^{(i)}}_{\omega_{(i)}^C} + \underbrace{\hat{\Phi}_a^{(i)}\hat{D}_E^{ab}\hat{\Phi}_b^{(i)}}_{\omega_{(i)}^D}}$$

(4) Determination of the reasons for an instability

$$\boxed{\begin{array}{l} \omega_{(i)} \leqslant 0, \omega_{(i)}^C < 0 \Rightarrow \text{material instability} \\ \omega_{(i)} \leqslant 0, \omega_{(i)}^D < 0 \Rightarrow \text{structural instability} \end{array}}$$

$(\mathbf{b} - \lambda_{(i)}^2\mathbf{1}) \cdot \mathbf{n}_{(i)} = \mathbf{0}$ as base vectors, the constitutive tensor $\mathscr{C}$ can be written as

$$\mathscr{C} = C_E^{ijkl}\mathbf{n}_i \otimes \mathbf{n}_j \otimes \mathbf{n}_k \otimes \mathbf{n}_l, \quad i,j,k,l = 1,2,3.$$

Note that $\mathbf{b} = \mathbf{F}\cdot\mathbf{F}^T$ and $\mathbf{C} = \mathbf{F}^T\cdot\mathbf{F}$ have the same eigenvalues. The evaluation of the strain energy function (1) leads with eqns (11) and (19) to

$$\boxed{C_E^{iiii} = \hat{C}_E^{ii} = 4\frac{\lambda_i^4}{J}\left[\frac{\partial^2 W}{\partial C_i \partial C_i} - p\frac{\partial^2 U}{\partial C_i \partial C_i}\right] = \sum_{r=1}^{n}[\mu_r(\alpha_r - 2)\lambda_i^{\alpha_r}] + p, \quad i = 1,2,3,}$$

where $C_i = \lambda_i^2$ denotes the principal values of the right Cauchy–Green tensor $\mathbf{C}$. Furthermore, we obtain

$$\boxed{C_E^{iijj} = \hat{C}_E^{ij} = 4\frac{\lambda_i^2\lambda_j^2}{J}\left[\frac{\partial^2 W}{\partial C_i \partial C_j} - p\frac{\partial^2 U}{\partial C_i \partial C_j}\right] = -p, \quad i,j = 1,2,3, \quad i \neq j.}$$

For the derivation of the remaining coefficients see Chadwick and Ogden (1971 a; b) or Ogden (1984). We have

$$\boxed{\begin{array}{ll} \frac{1}{2}(C_E^{1212} + C_E^{1221}) = \hat{C}_E^{44} = \frac{\lambda_1^2\lambda_2^2}{J}\frac{S^1 - S^2}{C_1 - C_2} = \frac{\sigma^1\lambda_2^2 - \sigma^2\lambda_1^2}{\lambda_1^2 - \lambda_2^2}, & \lambda_1 \neq \lambda_2, \\ \frac{1}{2}(C_E^{1212} + C_E^{1221}) = \hat{C}_E^{44} = \sum_{r=1}^{n}\frac{1}{2}(\hat{C}_E^{11} - \hat{C}_E^{12}), & \lambda_1 = \lambda_2 \end{array}} \tag{22}$$

with the principal values S^i of the second Piola–Kirchhoff stress tensor $\mathbf{S}$ and the principal values

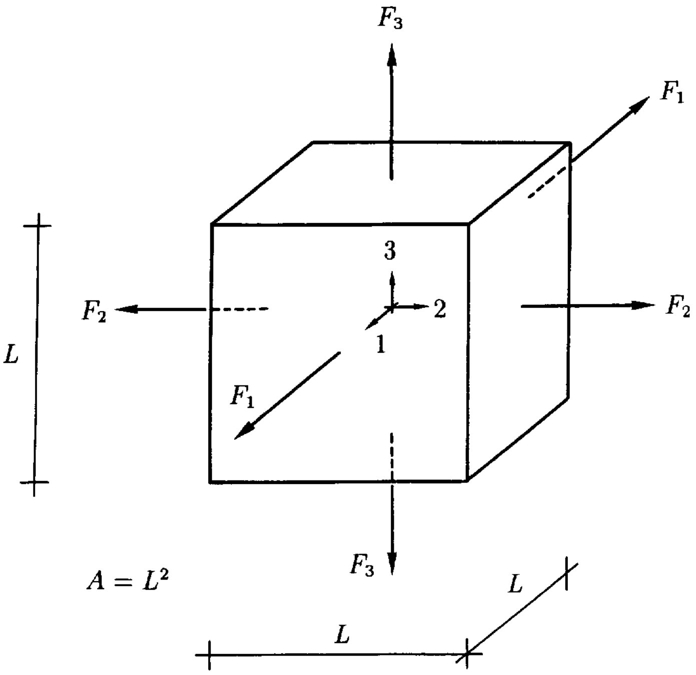


Fig. 1. Cube under triaxial tension.

$$\boxed{\sigma^i = \sum_{r=1}^{n} \mu_r \lambda_i^{\alpha_r} - p, \quad i = 1, 2, 3}$$

of the Cauchy stress tensor $\boldsymbol{\sigma} = 1/J(\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{\mathrm{T}})$. The coefficients $\hat{C}_E^{55}$ and $\hat{C}_E^{66}$ are calculated analogously, by exchanging the indices in eqn (22). For the coefficients of the "stress tangent tensor" $\mathscr{D}$, we obtain

$$\boxed{\begin{aligned} D_E^{iiii} &= \hat{D}_E^{ii} = \sigma^i, \quad i = 1, 2, 3 \\ D_E^{iijj} &= \hat{D}_E^{ij} = 0, \quad i, j = 1, 2, 3, \quad i \neq j \end{aligned}}$$

and

$$\boxed{\tfrac{1}{4}(\bar{D}_E^{1212} + \bar{D}_E^{2112} + \bar{D}_E^{1221} + \bar{D}_E^{2121}) = \hat{D}_E^{44} = \tfrac{1}{4}(\sigma^1 + 0 + 0 + \sigma^2).}$$

$\hat{D}_E^{55}$ and $\hat{D}_E^{66}$ have to be determined analogously. All other coefficients of $\mathscr{C}$ and $\mathscr{D}$ are equal to zero with respect to principal axes.

5.1. *Equilibrium*

The equilibrium condition for the cube in Fig. 1, Div $\mathbf{P} = 0$, yields, combined with the constitutive law, the principal first Piola–Kirchhoff stresses

$$P^i = \sum_{r=1}^{n} \mu_r \lambda_i^{\alpha_r - 1} - \frac{p}{\lambda_i} = \frac{F_i}{A} \tag{23}$$

Due to $F_1 = F_2 = F_3$ the equality $P^1 = P^2 = P^3$ holds.

We have to distinguish between three solution cases.

(1) $\lambda_1 = \lambda_2 = \lambda_3 = 1$

This first solution represents the primary solution path which is associated with the undeformed system. It has to be emphasized that this deformation state is not necessarily stress-free: eqn (23) yields

$$\boxed{P^i = \sum_{r=1}^{n} \mu_r - p,} \tag{24}$$

where the hydrostatic pressure remains variable.

(2) $\lambda_1 \neq \lambda_2$; $\lambda_3 \neq \lambda_2$

In this case the hydrostatic pressure can be determined. From $P^1 = P^2$ we obtain

$$p\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) = \sum_{r=1}^{n} \mu_r (\lambda_1^{\alpha_r - 1} - \lambda_2^{\alpha_r - 1})$$

$$\Rightarrow p = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \sum_{r=1}^{n} \mu_r (\lambda_1^{\alpha_r - 1} - \lambda_2^{\alpha_r - 1}). \tag{25}$$

The limit procedure $\lambda_1 \to \lambda_2$, for $\lambda_2 = 1$, yields

$$\lim_{\lambda_1 \to \lambda_2} p = \sum_{r=1}^{n} \mu_r (1 - \alpha_r)$$

and finally for all components P^i

$$P^i = \sum_{r=1}^{n} \mu_r \alpha_r. \tag{26}$$

Therefore, the so-called secondary load–deformation path ($\lambda_1 \neq \lambda_2$; $\lambda_3 = \lambda_2$) bifurcates from the primary path at the stress value (26). We further obtain, with the hydrostatic pressure (25),

$$\boxed{P^1 = \sum_{r=1}^{n} \mu_r \lambda_1^{\alpha_r - 1} - \frac{\lambda_2}{\lambda_2 - \lambda_1} \sum_{r=1}^{n} \mu_r (\lambda_1^{\alpha_r - 1} - \lambda_2^{\alpha_r - 1}) = P^2 = P^3} \tag{27}$$

as stress–deformation relation. Note that the preceding derivation holds analogously for $\lambda_1 \neq \lambda_2$; $\lambda_3 = \lambda_1$ and for cyclic permutations of the indices.

(3) $\lambda_1 \neq \lambda_2 \neq \lambda_3$

On the so-called tertiary solution path there are three distinct principal stretches λ_i. With eqns (25) and (27), which are also valid for three distinct λ_i, the condition

$$P^1 = \sum_{r=1}^{n} \mu_r \lambda_1^{\alpha_r - 1} - \frac{\lambda_2}{\lambda_2 - \lambda_1} \sum_{r=1}^{n} \mu_r (\lambda_1^{\alpha_r - 1} - \lambda_2^{\alpha_r - 1})$$

$$= \sum_{r=1}^{n} \mu_r \lambda_3^{\alpha_r - 1} - \frac{1}{\lambda_3} \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \sum_{r=1}^{n} \mu_r (\lambda_1^{\alpha_r - 1} - \lambda_2^{\alpha_r - 1}) = P^3 \tag{28}$$

has to be fulfilled. In the case of a Mooney–Rivlin material with $\mu_2 = -k\mu_1$, eqn (28) simplifies to

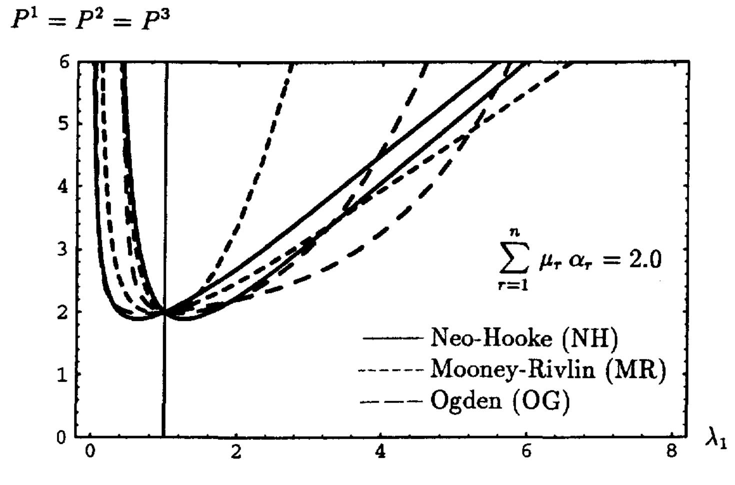
$$(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)\underbrace{(k(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) - 1)}_{A} = 0.$$

It is immediately clear that, for the neo-Hooke model, because of $k = 0 \Rightarrow A = -1$, a tertiary solution path does not exist. If one assumes that the tertiary solution path bifurcates from the secondary path $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$, then, in the general case ($k \neq 0$) the quantity A goes to zero at the intersection of the secondary and tertiary paths, so that

$$k = \frac{\lambda_1}{\lambda_1^3 + 2} \quad \text{or} \quad \lambda_1^3 - \frac{1}{k}\lambda_1 + 2 = 0 \tag{29}$$

at that point. For $k = 1/3$, the cubic equation (29) has the only solution $\lambda_1 = 1$. If $0 < k < 1/3$ holds, there are two physically relevant solutions. For $k > 1/3$ a tertiary solution path does not exist. In the case of an Ogden material, this investigation can only be carried out using a symbolic manipulation system like Mathematica [see Wolfram (1991)]. We will discuss the existence of a tertiary path for this material model in combination with the stability calculation.

The primary and secondary stress–deformation paths for three different material models are plotted in Fig. 2. The material parameters are chosen such that $\Sigma_{r=1}^{n} \mu_r \alpha_r = 2.0$ holds.



	NH	MR	OG
μ_1	1.0	0.8333	1.4911
α_1	2.0	2.0	1.3
μ_2	–	-0.1667	-0.02367
α_2	–	-2.0	-2.0
μ_3	–	–	0.00284
α_3	–	–	5.0

Fig. 2. Primary and secondary solution paths.

Therefore, the primary and secondary solution paths for all three material models intersect in one point, i.e. for $P^i = 2.0$. For every material model two secondary solution paths associated with $\lambda_1 \neq \lambda_2 = \lambda_3$ and $\lambda_1 = \lambda_2 \neq \lambda_3$ are depicted.

5.2. *Stability investigation*

In this section we investigate the stability behaviour of a cube for the Mooney–Rivlin and the Ogden material models. For the Mooney–Rivlin material model the results known from the literature (Ball and Schaeffer, 1983) can be verified, however, we will see that the criterion (18) excludes rigid body modes.

Mooney–Rivlin material. The Mooney–Rivlin material with

$$\mu_2 = -k\mu_1, \quad \alpha_1 = 2, \quad \alpha_2 = -2 \tag{30}$$

includes as a special case ($k = 0$) the neo-Hooke model ($\mu_2 = 0$).

First, we investigate the primary path $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The solution of the eigenvalue problem (21) yields the eigenvalues

$$\begin{aligned} \omega_1 &= 3k\mu_1 + \mu_1 - 2p = \underbrace{4k\mu_1 - p}_{\omega_1^C} + (\underbrace{-k\mu_1 + \mu_1 - p}_{\omega_1^P}), \\ \omega_2 &= 3k\mu_1 + \mu_1 + p = \underbrace{4k\mu_1 + 2p}_{\omega_2^C} + (\underbrace{-k\mu_1 + \mu_1 - p}_{\omega_2^P}), \end{aligned} \tag{31}$$

$$\omega_3 = \omega_2 = 2\omega_4 = 2\omega_5 = 2\omega_6. \tag{32}$$

The coefficients of the associated eigenvectors are

$$\{\hat{\Phi}^1\}_j = \frac{1}{\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \{\hat{\Phi}^2\}_j = \frac{1}{\sqrt{2}}\begin{Bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \{\hat{\Phi}^3\}_j = \frac{1}{\sqrt{2}}\begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\{\hat{\Phi}^4\}_j = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \{\hat{\Phi}^5\}_j = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}, \{\hat{\Phi}^6\}_j = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{33}$$

They are plotted in Fig. 3.

Due to the extra unknown in eqn (31), i.e. the hydrostatic pressure p, an additional equation is required. Since the determinant of the material deformation gradient $\mathbf{F}$ is, because of the incompressibility condition, equal to 1, we can write

$$\Delta J = \frac{\partial(\det \mathbf{F})}{\partial \mathbf{F}} : \Delta\mathbf{F} = \mathbf{F}^{-\mathrm{T}} : \Delta\mathbf{F} = \frac{\Delta\lambda_1}{\lambda_1} + \frac{\Delta\lambda_2}{\lambda_2} + \frac{\Delta\lambda_3}{\lambda_3} = 0. \tag{34}$$

For the coefficients $\{\hat{\Phi}^{(i)}\}_j$ of the eigenvector this yields

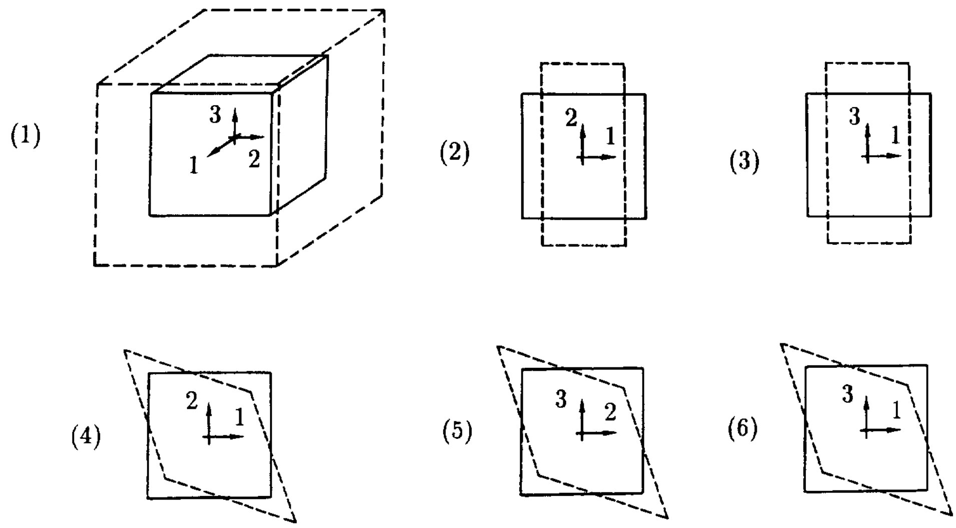


Fig. 3. Eigenmodes.

$$\hat{\Phi}_1^{(i)} + \hat{\Phi}_2^{(i)} + \hat{\Phi}_3^{(i)} = 0,$$

so that the eigenform $\{\Phi^1\}_i$ is kinematically not possible. Therefore, we have a singular solution for

$$\omega_2 = 0 \Rightarrow p_{\text{crit}} = -\mu_1(1+3k),$$

from which follows with eqn (24)

$$\boxed{\sigma^i_{\text{crit}} = P^i_{\text{crit}} = \mu_1 - k\mu_1 + \mu_1(1+3k) = 2\mu_1(1+k).}$$

Referring to eqn (26), which simplifies for the Mooney–Rivlin-material to $P^i = 2\mu_1(1+k) = P^i_{\text{crit}}$, it is immediately clear that the point of intersection between the primary and secondary solution path marks the first singularity of the cube under tension.

From eqns (31) and (32), we obtain, with $l = 2, 3, 4, 5, 6$

$$\boxed{\omega_l = 0, \quad \omega_l^C < 0 \Rightarrow \text{material instability}}$$

It is essential that this material instability is characterized by a fivefold zero eigenvalue. In contrast to earlier work (Rivlin, 1974; Sawyers, 1976; Ball and Schaeffer, 1983), where only the stretch modes were mentioned, here three shear eigenmodes are also obtained. Due to the fact that the stability investigation is carried out locally, we have simultaneously, in every material point of the body, the same situation : there is an infinite number of linear combinations of the eigenmodes (33) which also may become effective. For a global investigation of the cube, which could be carried out by means of finite elements, we would obtain a large cluster of zero eigenvalues for the load level associated with σ_{crit}. Note that this effect can only be demonstrated if a finite element formulation is used which does not exhibit locking behaviour. The material stability behaviour stands, therefore, in crucial contrast to structural stability behaviour, where only by a global stability investigation instabilities can be detected.

Remark 2

For the natural state ($\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\sigma^i = 0$) the material tensor of linear elasticity (small strains, linear stress–strain relation) is recovered from the constitutive tensor

$\mathscr{C}$. Note that, due to $p_{\text{lin}} = -\sigma^i + \mu = 0 + \mu \neq p_{\text{crit}}$, the solution is always unique in linear elasticity theory.

Remark 3

In contrast to the present work Rivlin (1974) and Ball and Schaeffer (1983) used the stability criterion (8). With the well-known relation [see Ogden (1984), p. 342]

$$A_E^{ijkl} = \frac{J}{\lambda_i \lambda_k} C_E^{ijkl} + \frac{J}{\lambda_i^2} \sigma^i \delta^{ik} \delta^{jl}, \quad i,j,k,l = 1,2,3$$

one derives for $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$$\begin{aligned}
\bar{A}_E^{iiii} &= \sum_{r=1}^{n} \mu_r(\alpha_r - 2) + p + \sum_{r=1}^{n} \mu_r - p = \sum_{r=1}^{n} \mu_r(\alpha_r - 1) &&= \beta, \\
\bar{A}_E^{iijj} &= -p &&= \gamma, \\
\bar{A}_E^{ijij} &= \sum_{r=1}^{n} \frac{\mu_r}{2}(\alpha_r - 2) + p + \sum_{r=1}^{n} \mu_r - p = \sum_{r=1}^{n} \frac{1}{2} \mu_r \alpha_r &&= \delta, \\
\bar{A}_E^{ijji} &= \sum_{r=1}^{n} \tfrac{1}{2} \mu_r(\alpha_r - 2) + p = \varepsilon.
\end{aligned}$$

For clarity, we write the eigenvalue problem (7) in matrix form:

$$\begin{bmatrix}
\beta & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma & \beta & \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma & \gamma & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & \varepsilon & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varepsilon & \delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & \varepsilon & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \varepsilon & \delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & \varepsilon \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon & \delta
\end{bmatrix}
\begin{Bmatrix}
\tilde{\Phi}_{11}^{(i)} \\ \tilde{\Phi}_{22}^{(i)} \\ \tilde{\Phi}_{33}^{(i)} \\ \tilde{\Phi}_{12}^{(i)} \\ \tilde{\Phi}_{21}^{(i)} \\ \tilde{\Phi}_{23}^{(i)} \\ \tilde{\Phi}_{32}^{(i)} \\ \tilde{\Phi}_{31}^{(i)} \\ \tilde{\Phi}_{13}^{(i)}
\end{Bmatrix}
= \tilde{\omega}_{(i)}
\begin{Bmatrix}
\tilde{\Phi}_{11}^{(i)} \\ \tilde{\Phi}_{22}^{(i)} \\ \tilde{\Phi}_{33}^{(i)} \\ \tilde{\Phi}_{12}^{(i)} \\ \tilde{\Phi}_{21}^{(i)} \\ \tilde{\Phi}_{23}^{(i)} \\ \tilde{\Phi}_{32}^{(i)} \\ \tilde{\Phi}_{31}^{(i)} \\ \tilde{\Phi}_{13}^{(i)}
\end{Bmatrix}. \tag{35}$$

The results of Rivlin (1974) and Ball and Schaeffer (1983) can now easily be verified. They obtained, with respect to eqns (31) and(32), three additional eigenvalues

$$\tilde{\omega}_7 = \tilde{\omega}_8 = \tilde{\omega}_9 = \sum_{r=1}^{n} \mu_r - p.$$

The coefficients of the associated eigentensors are

$$[\tilde{\Phi}^7]_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\tilde{\Phi}^8]_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$[\tilde{\Phi}^9]_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{36}$$

which represent rigid body rotations. Surprisingly, the eigenvalues $\tilde{\omega}_8 = \tilde{\omega}_8 = \tilde{\omega}_9$ go to zero for $p = \Sigma_{r=1}^{n}\mu_r$, i.e. for the natural state! Evidently, this singularity cannot be physically reasonable since the natural state is identified with the unloaded cube. Therefore, rigid body motions as eigenmodes should be excluded from the eigenvalue problem which requires the use of the stability criterion (18) instead of (8). Finally, this justifies the definition of material instability in Section 4.

Note that in linear elasticity this case is automatically precluded by using the constitutive tensor with respect to the symmetric strain measure $\mathbf{E}_{lin} = \frac{1}{2}$ (Grad $\mathbf{u}$+Grad$^{\mathrm{T}}\mathbf{u}$) [$\mathbf{u}$ displacement vector], which does not contain the linearized rigid body modes.

It remains to investigate the stability behaviour on the secondary deformation path. Now we have two equal principal stretches $\lambda_1 = \lambda_2 = \lambda$. The third principal stretch follows from $\lambda_1\lambda_2\lambda_3 = 1 \Rightarrow \lambda_3 = 1/\lambda^2$. Only the eigenvalues

$$
\begin{aligned}
\omega_2 &= \frac{\mu_1}{\lambda^4}(2k\lambda^2 - \lambda^3 - k\lambda^5 + \lambda^6 - k\lambda^8) \\
&= \underbrace{\frac{\mu_1}{\lambda_4}(2k\lambda^2 - 2\lambda^3 - 2k\lambda^5 - 2k\lambda^8)}_{\omega_2^C} + \underbrace{\frac{\mu_1}{\lambda_4}(\lambda^3 + k\lambda^5 + \lambda^6 + k\lambda^8)}_{\omega_2^D}
\end{aligned}
$$

and

$$\omega_4 = \tfrac{1}{2}\omega_2$$

with the eigenvectors

$$
2\{\hat{\Phi}^2\}_j = \frac{1}{\sqrt{2}}\begin{Bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \{\hat{\Phi}^4\}_j = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}
$$

can become negative. $\omega_2 = \omega_4 = 0$ yields the bifurcation condition

$$\boxed{k = \frac{\lambda}{\lambda^3 + 2},}$$

which coincides, as expected, with eqn (29). The function $k = \hat{k}(\lambda)$ is plotted in Fig. 4. Let us investigate, for example, the material parameter ratio $k = 0.2$; the sign of ω changes for $\lambda = 0.415$, $\lambda = 1$ and $\lambda = 2$, where $\omega = 0$ for $\lambda = 1$ indicates the bifurcation from the primary solution path into the secondary path. Therefore, we have bifurcations from the secondary into the tertiary path for $\lambda = 0.415$ and $\lambda = 2$. Since $\omega^C < 0$ holds for $\lambda > 0.2$, it is evident that we deal with material instabilities for $k = 0.2$.

Only for $0 < k \leqslant \frac{1}{3}$ does a solution $\lambda_1 \neq \lambda_2 \neq \lambda_3$ exist. This result correlates with the one of Ball and Schaeffer (1983), which was derived by equilibrium considerations (see also Section 5.1). If the bifurcation from the secondary into the tertiary path exists, it is always a material stability, since with $l = 2, 4$

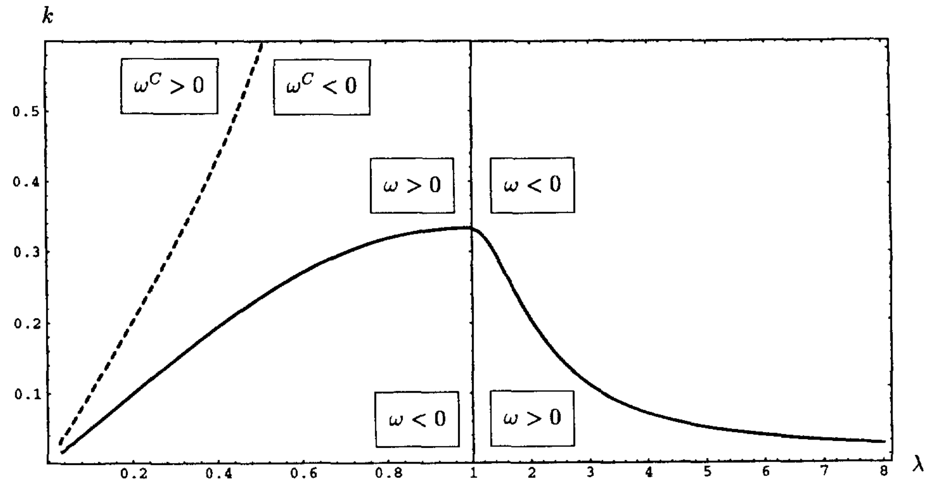


Fig. 4. Mooney–Rivlin material: material parameter ratio $k = -\frac{\mu_2}{\mu_1}$ as a function of λ.

$$\boxed{\omega_l = 0, \quad \omega_l^C < 0}$$

is valid. This singularity is indicated by a double zero eigenvalue. It is associated with a stretch and shear mode.

Ogden material. We now investigate the six-parameter model introduced by Ogden. Ogden (1972) determined the material parameters for a specific rubber

$$\begin{aligned} \mu_1 &= 6.3 & \alpha_1 &= 1.3 \\ \mu_2 &= -0.1 & \alpha_2 &= -2.0 \\ \mu_3 &= 0.012 & \alpha_3 &= 5.0 \end{aligned} \tag{37}$$

which correlate very well with the experiments of Treloar (1944) for 8% sulphur rubber. Treloar (1944) chose this kind of rubber because it shows highly reversible elastic behaviour and is free of crystallization up to ca. 400% strain. Therefore, the usual assumptions made in finite elasticity theory apply very well.

It is evident that changes of the material parameters could influence the stability behaviour as well. To investigate the dependence of stability behaviour on the material parameters, we introduce the material parameter ratios

$$\mu_2 = -k_2\mu_1 \quad \mu_3 = k_3\mu_1.$$

With the help of Mathematica one can show that

$$\det \hat{T}_E^{ij} = \underbrace{(\hat{\Phi}_i^1 \hat{T}_E^{ik} \hat{\Phi}_k^1)}_{\omega_1} \cdot \underbrace{(\hat{\Phi}_i^2 \hat{T}_E^{ik} \hat{\Phi}_k^2)}_{\omega_2} \cdot \ldots \cdot \underbrace{(\hat{\Phi}_i^6 \hat{T}_E^{ik} \hat{\Phi}_k^6)}_{\omega_6} = \omega_1\omega_2\omega_3\omega_4\omega_5\omega_6,$$

where the eigenvectors (33) have been used. Thus, interestingly, the eigenvectors are independent of the material model for this example. This substantially simplifies the calculation of the eigenvalues and the eigenvalue splitting with

$$\omega^C_{(i)} = \hat{\Phi}^{(i)}_a \hat{C}^{ab}_E \hat{\Phi}^{(i)}_b \quad \text{and} \quad \omega^D_{(i)} = \hat{\Phi}^{(i)}_a \hat{D}^{ab}_E \hat{\Phi}^{(i)}_b .$$

On the primary solution path, we have the same stability behaviour for the Ogden model as for the Mooney–Rivlin model, since the existence of the first singularity with

$$P^i_{\text{crit}} = \sum_{r=1}^{n} \mu_r \alpha_r$$

does not depend on the material model. In spite of this fact this singularity is caused by the negative definiteness of the constitutive tensor $\mathscr{C}$. This represents an exceptional case in finite elasticity, since material instabilities usually depend on the choice of the material model. But, we have the special case here that in the undeformed state the material models considered in this paper coincide.

More interesting is the investigation of the secondary solution path. In Fig. 5 (a)–(c) and Fig. 6 (a)–(b), the contours $\omega_2 = \omega_4 = 0$ and $\omega^C_2 = \omega^C_4 = 0$ are plotted for $\lambda_1 = \lambda_2 = \lambda$; $\lambda_3 = 1/\lambda^2$. Although for every calculation, i.e. for every figure, only one material parameter has been varied, the strong influence of slight variations of the material parameters on the stability behaviour is clear. In contrast to the Mooney–Rivlin material there can be not only one or three changes for the sign of $\omega = \omega_2$, but also two, four or five. Because of

$$\boxed{\omega_{2,4} = 0, \quad \omega^C_{2,4} < 0,}$$

the contour $\omega^C_2 = \omega^C_4 = 0$ depicts that we deal in every case with material instabilities, which, due to the strong material parameter dependence, was expected. Furthermore, it is clear that the values (37) [leading to $k_2 = 0.01587$, $k_3 = 0.0019$] lie in the positive area of ω. Thus, for the parameters of this special material there is no tertiary load–deformation path.

In Fig. 7 (a) and (b), the curves for the special cases $k_3 = 0$ and $k_2 = 0.2$ are plotted. For $\alpha_1 = 2$ and $\alpha_2 = -2$, we obtain as a special case the Mooney–Rivlin model. The dependence on k_2 for $k_3 = 0$ and $\alpha_1 = 2$, $\alpha_2 = -2$ has already been depicted in Fig. 4. It is demonstrated that four-parameter models allow only one, two or three changes for the sign of ω. This leads to the expected statement that the complexity of the material stability behaviour increases with the number of material parameters.

6. CONCLUSIONS

In the present paper, it is demonstrated that material instabilities in finite elasticity are caused by the negative definiteness of the material tensor $\mathscr{C}$. This information has been used to develop an easily applicable method, called eigenvalue splitting, which allows the distinguishing of material instabilities from structural instabilities like buckling of shells and plates. The essential results of the preceding investigation of the cube under triaxial tension are summarized in Table 2.

It is observed that the material stability behaviour is characterized by multiple zero eigenvalues. This clustering of zero eigenvalues occurs simultaneously in every point of the structure. Such behaviour stands in contrast to structural stability behaviour, where, in spite of a primary homogeneous stress state, secondary paths are associated only with inhomogeneous stress states. In the latter case, clustering of zero eigenvalues might still occur for a special choice of boundary conditions and geometry; but such zero eigenvalues are obtained from the summary of contributions of the whole structure, which can have positive or negative values.

The eigenvalue splitting method presented in this work has been developed in the context of the Ogden model. On this basis, earlier results obtained for the Mooney–Rivlin model could be recovered as special cases, facilitating validation of the method. This method, however, is not in any way restricted to the Ogden model.

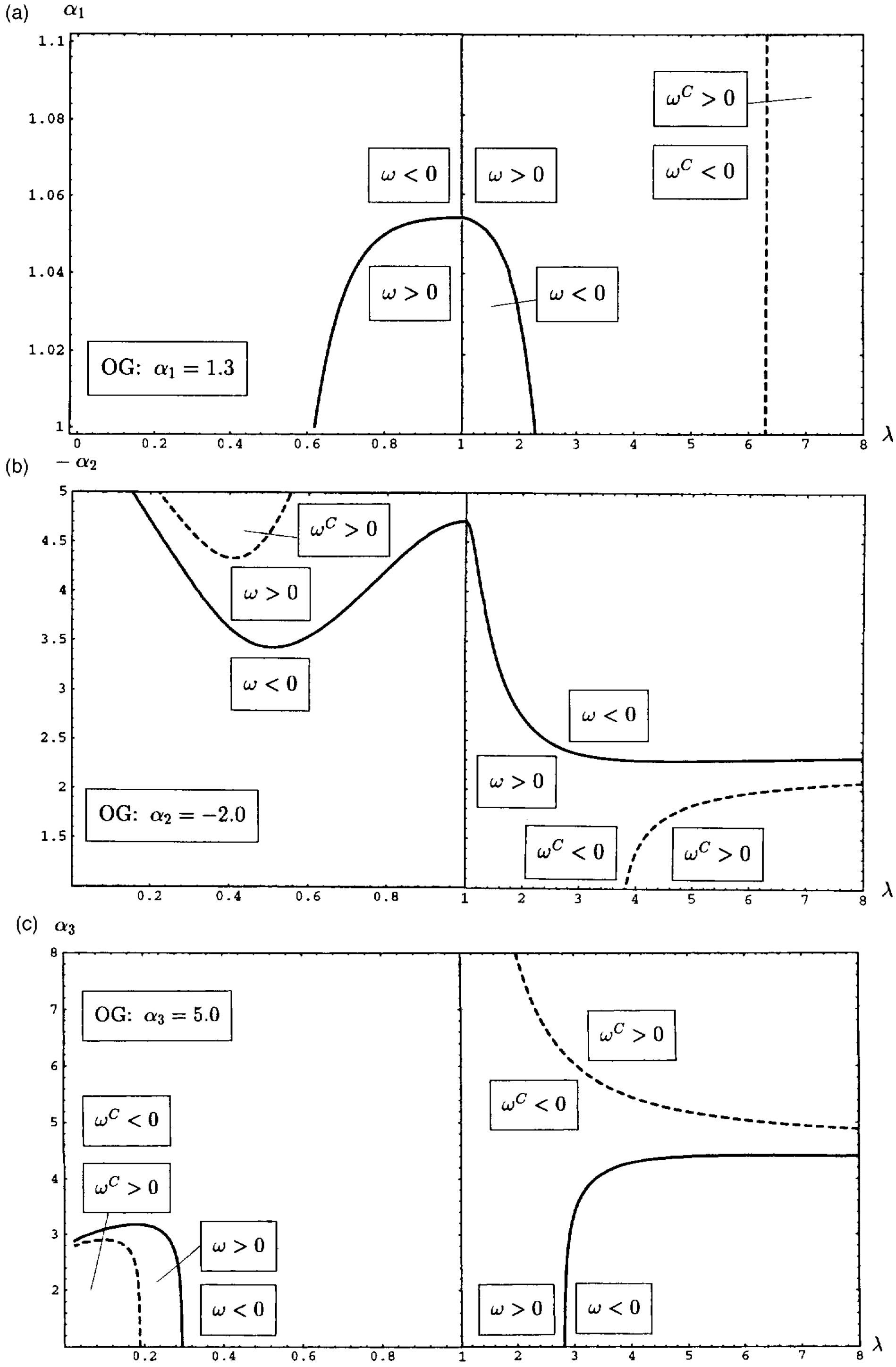


Fig. 5. Six-parameter model: functions $\omega = 0$ and $\omega_C = 0$ dependent on (a) material parameter α_1 and λ, (b) material parameter $-\alpha_2$ and λ, (c) material parameter α_3 and λ.

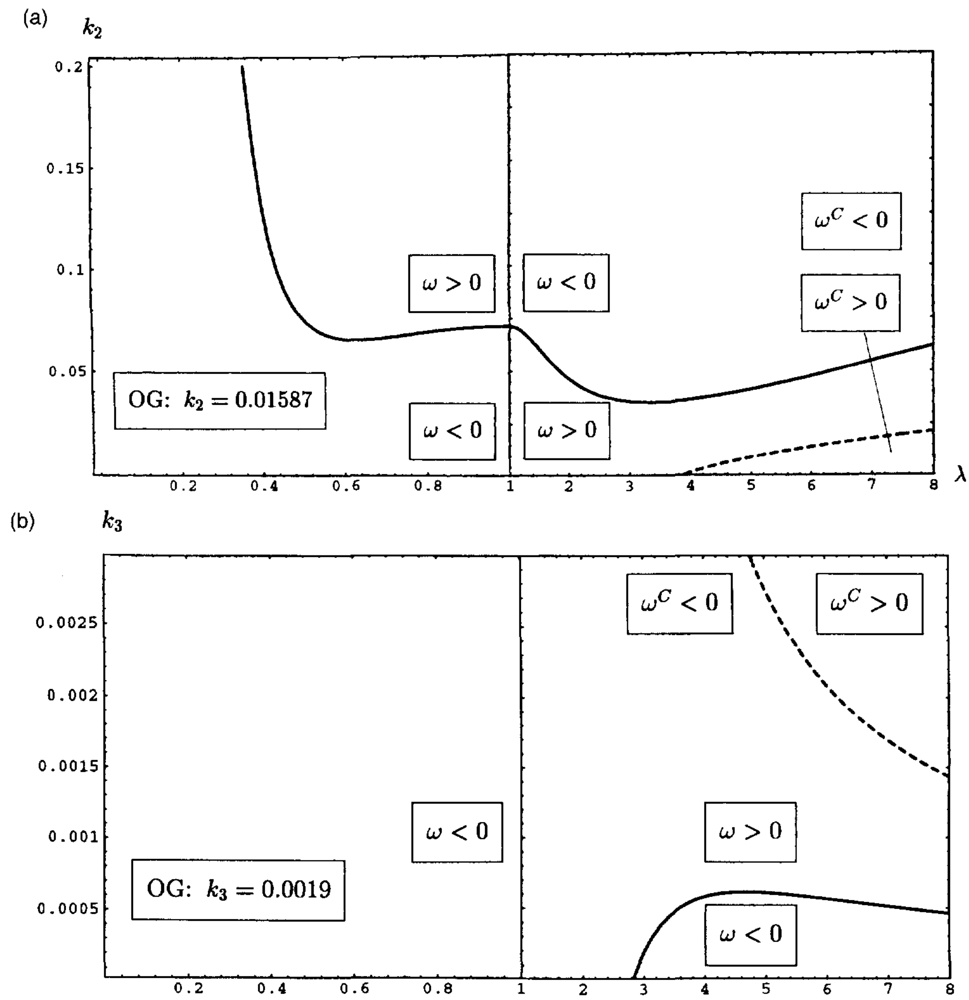


Fig. 6. Six-parameter model: functions $\omega = 0$ and $\omega_C = 0$ dependent on (a) material parameter k_2 and $-\mu_2/\mu_1$ and λ, (b)material parameter ratio $k_3 = -\mu_3/\mu_1$ and λ.

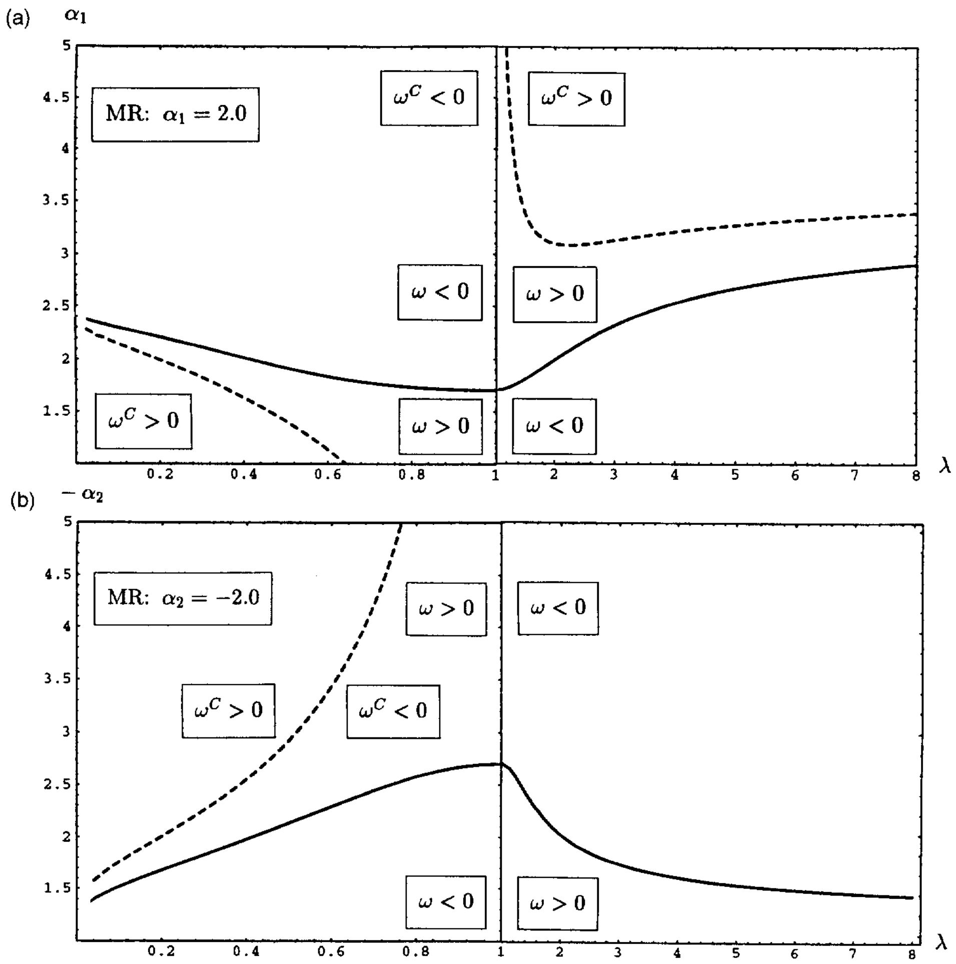


Fig. 7. Four-parameter model: functions $\omega = 0$ and $\omega_C = 0$ dependent on (a) material parameter α_1 and λ, (b) material parameter $-\alpha_2$ and λ.

Table 2. Essential results

(I)	Equilibrium Primary solution path: $\lambda_1 = \lambda_2 = \lambda_3 = 1$ • independent of the chosen material model Secondary solution path: $\lambda_1 = \lambda_2 \neq \lambda_3$ • quantitative influence of the material parameters • existence for every material model Tertiary solution path: $\lambda_1 \neq \lambda_2 \neq \lambda_3$ • existence strongly dependent on the material parameters
(II)	Stability investigation Bifurcation from the primary solution path: • fivefold zero eigenvalue • eigenforms: two stretch modes, three shear modes • material instability Bifurcation from the secondary solution path: • twofold zero eigenvalue • eigenforms: one stretch mode, one shear mode • material instability

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